Lecture 25: Hypercontractivity and Parity over Large Sets

Norm

- For $p \geqslant 1$, let us define the p-norm of a function

$$\|f\|_p = \left(\frac{1}{N} \sum_{x \in \{0,1\}^n} |f(x)|^p \right)^{1/p}$$

- We can use Jensen's inequality to prove that $\|f\|_p \leqslant \|f\|_q$ for any function f, when $p < q$, and equality holds if and only if $|f|$ is a constant function

Basic Observations

- We can show, using Jensen's inequality, that $\left\| T_\rho(f) \right\|_p \leqslant \|f\|_p$ (Intuition: because the noise operator smoothens the function)
- That is, "$T_\rho(\cdot)$ contracts f"
- By monotonicity of norm, we can conclude that $\left\| T_\rho(f) \right\|_p \leqslant \left\| T_\rho(f) \right\|_q$, for $p \leqslant q$
- So, we summarize the above discussion using the following picture

$$\begin{array}{ccc} \left\| T_\rho(f) \right\|_p & \leqslant & \|f\|_p \\ \rotatebox{90}{$\geqslant$} & & \\ \left\| T_\rho(f) \right\|_q & & \end{array}$$

- But how does $\|f\|_p$ relate to $\left\| T_\rho(f) \right\|_q$?

Hypercontractivity Theorem

Theorem (Hypercontractivity Theorem)

For $1 \leqslant p \leqslant q$ we have

$$\left\| T_\rho(f) \right\|_q \leqslant \| f \|_p ,$$

for all $0 \leqslant \rho \leqslant \sqrt{(p-1)/(q-1)}$.

- Intuitively, the hypercontractivity theorem states that even the q-th norm of $T_\rho(f)$ is smaller than the p-th norm of f, if q is slightly larger than p.
- The tightest inequality is obtained for $\rho = \sqrt{(p-1)/(q-1)}$

Special Case: $q = 2$

- Suppose $p = 1 + \delta$ and $q = 2$, then the hypercontractivity theorem states that

$$\left\| T_\rho(f) \right\|_2 \leqslant \|f\|_{1+\delta},$$

  for $\rho = \sqrt{\delta}$

- By Parseval's identity, we have

$$\left\| T_\rho(f) \right\|_2^2 = \sum_{S \in \{0,1\}^n} \delta^{|S|} \widehat{f}(S)^2$$

- So, we conclude the following result

$$\sum_{S \in \{0,1\}^n} \delta^{|S|} \widehat{f}(S)^2 \leqslant \|f\|_{1+\delta}^2$$

- Comment: Proving the hypercontractivity theorem for $1 \leqslant p \leqslant q = 2$ suffices to prove the general hypercontractivity theorem presented above

Application: KKL Lemma

- KKL stands for Kahn-Kalai-Linial
- Suppose $f\colon \{0,1\}^n \to \{-1,0,+1\}$
- Then, we have $\|f\|_p = \mathbb{P}[f \neq 0]^{1/p}$
- By direct application of the hypercontractivity theorem, we can conclude that

Lemma (KKL Lemma)

$$\sum_{S\in\{0,1\}^n} \delta^{|S|}\widehat{f}(S)^2 \leqslant \mathbb{P}[f \neq 0]^{2/(1+\delta)}$$

- Intuition: The left-hand side is dominated by the Fourier coefficients associated with "small-weight S." So, the inequality states that the "total mass associated with the Fourier coefficients of small-weight S" is much smaller than the probability of "encountering f." Equivalently, a "small support f" has low "total mass" on the Fourier coefficients of small S.

Application: Parities of Large Sets is Unpredictable I

- Suppose $A \subseteq \{0,1\}^n$ and $\mathbf{1}_{\{A\}}$ is the indicator function of the set A
- For $S \in \{0,1\}^n$, define

$$\beta_S = \frac{1}{|A|} \sum_{x \in A} \chi_S(x)$$

  Intuitively, the quantity β_S represents "how random the set A appears" when we perform the test χ_S. Smaller β_S implies more random A appears.

- We can perform the following manipulation

$$\beta_S = \frac{1}{|S|} \sum_{x \in A} \chi_S(x) = \frac{1}{|A|} \sum_{x \in \{0,1\}^n} \mathbf{1}_{\{A\}}(x)\chi_S(x) = \frac{N}{|A|} \langle \mathbf{1}_{\{A\}}, \chi_S \rangle = \frac{N}{|A|} \widehat{\mathbf{1}_{\{A\}}}(S)$$

Application: Parities of Large Sets is Unpredictable II

- Our goal is to study the following quantity

$$\sum_{S\in\{0,1\}^n : |S|=k} \beta_S^2$$

  Note that this quantity is the cumulative measure of randomness of all χ_S test such that $|S| = k$

- Now, we can use the expression of β_S to obtain

$$\sum_{S\in\{0,1\}^n : |S|=k} \beta_S^2 = \frac{N^2}{|A|^2} \sum_{S\in\{0,1\}^n : |S|=k} \widehat{\mathbf{1}_{\{A\}}}(S)^2$$

- The KKL Lemma provides us the perfect tool to upper-bound the right hand side. For any $\delta \in [0, 1]$, we have

$$\sum_{S\in\{0,1\}^n : |S|=k} \delta^k \widehat{f}(S)^2 \leqslant \sum_{S\in\{0,1\}^n} \delta^{|S|} \widehat{f}(S)^2 \leqslant \mathbb{P}\left[f \neq 0\right]^{2/(1+\delta)}$$

- So, we obtain

$$\begin{aligned}
\sum_{S\in\{0,1\}^n\colon |S|=k} \beta_S^2 &= \frac{N^2}{|A|^2} \sum_{S\in\{0,1\}^n\colon |S|=k} \widehat{\mathbf{1}_{\{A\}}}(S)^2 \\
&\leqslant \frac{N^2}{|A|^2} \cdot \frac{1}{\delta^k} \cdot \mathbb{P}\left[\mathbf{1}_{\{A\}} \neq 0\right]^{2/(1+\delta)} \\
&= \frac{N^2}{|A|^2} \cdot \frac{1}{\delta^k} \cdot \left(\frac{|A|}{N}\right)^{2/(1+\delta)} \\
&= \left(\frac{N}{|A|}\right)^{2\left(1-\frac{1}{1+\delta}\right)} \cdot \frac{1}{\delta^k} = \left(\frac{N}{|A|}\right)^{\frac{2\delta}{1+\delta}} \cdot \frac{1}{\delta^k} \\
&\leqslant \left(\frac{N}{|A|}\right)^{2\delta} \cdot \frac{1}{\delta^k}
\end{aligned}$$

- Now, our objective is to find $\delta \in [0,1]$ that minimizes the right hand side expression. This part is left as an exercise.
- At the end, for this value of δ, we shall have

$$\binom{n}{k}^{-1} \sum_{S \in \{0,1\}^n : |S| = k} \beta_S^2 = O\left(1 - \frac{a}{n}\right)^k,$$

where $|A| = 2^a$. That is, the average bias is exponentially small. This bound is also (essentially) tight.